Alternative linear structures associated with regular Lagrangians. Weyl quantization and the Von Neumann uniqueness theorem

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Abstract. We discuss how the existence of a regular Lagrangian description on the tangent bundle TQ of some configuration space Q allows for the construction of a linear structure on TQ that can be considered as "adapted" to the given dynamical system. The fact then that many dynamical systems admit alternative Lagrangian descriptions opens the possibility to use the Weyl scheme to quantize the system in different non equivalent ways, "evading", so to speak, the von Neumann uniqueness theorem.

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1. Introduction

The main purpose of this Note is to discuss how the existence of a regular Lagrangian description $\mathcal{L}$ on the tangent bundle TQ of some configuration space Q, can allow, under suitable assumptions that will be discussed shortly below, for the "dynamical" construction of a linear structure on TQ that can be considered as "adapted" to the given dynamical system. If and when this is possible, one obtains a new action of the group $\mathbb{R}^{2n}$ ($n = \dim Q$) on TQ and, as will be shown, the Lagrangian two-form $\omega_{\mathcal{L}}$ can be put explicitly in canonical Darboux form. One can then follow the Weyl procedure [1] to quantize the dynamics, by realizing the associated Weyl system on the Hilbert space of square-integrable functions on a suitable Lagrangian submanifold of TQ.

The fact that many dynamical systems admit genuinely alternative§ Lagrangian descriptions [2] poses an interesting question, namely: assume that a given dynamical system admits alternative Lagrangian descriptions with more than one regular Lagrangian. According to what has been outlined above, one will possibly obtain different actions (realizations) of the group $\mathbb{R}^{2n}$ on TQ that in general will not be linearly related. Then, it will be possible to quantize "à la" Weyl the system in two different ways, thereby obtaining different Hilbert space structures on spaces of square-integrable functions on different Lagrangian submanifolds (actually what appears as a Lagrangian submanifold in one scheme need not be such in the other. Moreover, the Lebesgue measures will be different in the two cases). The occurrence of this situation seems then to offer the possibility of, so-to-speak,"evading" the von Neumann theorem [3] and this is one of the topics to be discussed in this Note.

Before embarking in the general discussion, we recall here some known facts about the possibility of defining alternative (i.e. not linearly related) linear structures on a vector space and/or of using the linear structure of a vector space to endow with a linear structure manifolds that are related to the given vector space.

2. Alternative linear structures

Let E be a (real or complex) linear vector space. A (not necessarily linear) diffeomorphism:

$$\phi : E \leftrightarrow M \tag{1}$$

with M a manifold (possibly $M = E$) allows us to "import" a linear structure from E to M. In particular, if $M = E$, we can define an alternative linear structure on E itself. To do so, we proceed by defining:

- Addition of $u, v \in M$ as:

$$u +_{(\phi)} v =: \phi(\phi^{-1}(u) + \phi^{-1}(v)). \tag{2}$$

§ I.e. not differing merely by the addition of a total time derivative to the Lagrangian.

- Multiplication by a scalar $\lambda \in \mathbb{R}$ or $\mathbb{C}$ of $u \in M$ as:

$$\lambda \cdot_{(\phi)} u =: \phi\left(\lambda\phi^{-1}(u)\right). \tag{3}$$

These operations have all the usual properties of addition and multiplication by a scalar. In particular:

$$(\lambda\lambda') \cdot_{(\phi)} u = \lambda \cdot_{(\phi)} \left(\lambda' \cdot_{(\phi)} u\right) \tag{4}$$

and:

$$\left(u +_{(\phi)} v\right) +_{(\phi)} w = u +_{(\phi)} \left(v +_{(\phi)} w\right). \tag{5}$$

Indeed, e.g.:

$$\lambda \cdot_{(\phi)} \left(\lambda' \cdot_{(\phi)} u\right) = \phi\left(\lambda\phi^{-1}\left(\lambda' \cdot_{(\phi)} u\right)\right) = \phi\left(\lambda\lambda'\phi^{-1}(u)\right) = (\lambda\lambda') \cdot_{(\phi)} u \tag{6}$$

which proves (4), and similarly for (5). ■

To every linear structure there is associated in a canonical way a *dilation* (or Liouville) field Δ which is the infinitesimal generator of dilations (and in fact defines [4] the linear structure). Therefore, in the framework of the new linear structure, it makes sense to consider the mapping:

$$\Psi : M \times \mathbb{R} \to M \tag{7}$$

via:

$$\Psi(u,t) =: e^t \cdot_{(\phi)} u =: u(t), \tag{8}$$

i.e.:

$$u(t) = \phi\left(e^t\phi^{-1}(u)\right). \tag{9}$$

Property (4) ensures that:

$$\Psi\left(u(t'), t\right) = \Psi\left(u, t+t'\right), \tag{10}$$

i.e. that (8) is indeed a one-parameter group. Then, the infinitesimal generator of the group is defined as:

$$\Delta(u) = \left[\frac{d}{dt}u(t)\right]_{t=0} = \left[\frac{d}{dt}\phi\left(e^t\phi^{-1}(u)\right)\right]_{t=0}. \tag{11}$$

Explicitly, in components:

$$\Delta = \Delta^i \frac{\partial}{\partial u^i} \tag{12}$$

and:

$$\Delta^i = \left[\frac{\partial\phi^i(w)}{\partial w^j}w^j\right]_{w=\phi^{-1}(u)}. \tag{13}$$

In other words, if we denote by $\Delta_0 = w^i\partial/\partial w^i$ the Liouville field associated with the standard linear structure on E:

$$\Delta = \phi_*\Delta_0, \tag{14}$$

where ϕ_* denotes, as usual, the push-forward.

Remark. If $M = E$ and ϕ is a linear (and invertible) map, then (13) yields: $\Delta^i = u^i$, i.e.:

$$\phi_* \Delta_0 = \Delta_0. \tag{15}$$

Examples.

We shall discuss here a couple of examples. Other simple examples are described in the Appendix.

- As a first example, consider $T^*\mathbb{R}$ with coordinates (q, p) and the standard symplectic form $\omega = dq \wedge dp$. The linear structure is defined by the dilation (Liouville) field:

  $$\Delta = q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} \tag{16}$$

  and is such that:

  $$i_\Delta \omega = qdp - pdq =: 2\theta \tag{17}$$

  and: $\omega = d\theta$. Another relevant structure that can be constructed is the complex structure, that is defined by the $(1,1)$ tensor field:

  $$J = dp \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial p}, \tag{18}$$

  which satisfies $J^2 = -\mathbb{I}$ (the identity) and, being constant, has a vanishing Nijenhuis tensor [5]: $N_J = 0$. Notice that:

  $$J \circ \omega = g, \tag{19}$$

  where g is the $(2,0)$ tensor:

  $$g = dq \otimes dq + dp \otimes dp, \tag{20}$$

  i.e. a (Euclidean) metric tensor, and: $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$.

  Remark. In this way we have defined three structures on a cotangent bundle (actually on the cotangent bundle of a vector space), namely a symplectic structure, a complex structure and a metric tensor. It should be clear from, e.g., eq. (19) that, given, say, g, we can define in turn the complex structure as:

  $$J = g \circ \omega^{-1}. \tag{21}$$

  In other words, the three structures are not independent: given any two of them the third one is defined in terms of the previous ones [6].

  We recall now [2] that with any $(1,1)$ tensor field S with vanishing Nijenhuis tensor one can associate an antiderivation d_S of degree one satisfying‖ $d_S^2 = 0$ and that

‖ As a consequence of the Nijenhuis condition $N_S = 0$.

acts in particular on functions as: $(d_S f)(X) = df(S \cdot X)$ with X any vector field, or:

$$d_S f = \widehat{S} \cdot df, \tag{22}$$

with $\widehat{S}$ denoting the action of S on forms. Then it is easy to prove that:

$$\theta = \frac{1}{2} d_J \left(\frac{1}{2} \left(p^2 + q^2\right) \right) \tag{23}$$

and hence also:

$$\omega = \frac{1}{2} d d_J \left(\frac{1}{2} \left(p^2 + q^2\right) \right). \tag{24}$$

Consider now the dynamics of the $1D$ harmonic oscillator that, in appropriate units, is described, as is well known, by the vector field:

$$\Gamma = p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p}, \tag{25}$$

which is ω-Hamiltonian: $i_\Gamma \omega = dH$ with Hamiltonian: $H = \left(q^2 + p^2\right)/2$. Notice that:

$$\Gamma = J(\Delta). \tag{26}$$

Consider the the nonlinear change of coordinates [7]: $(q, p) \to (Q, P)$ with:

$$\begin{aligned} Q &= q\,(1 + f(H)) \\ P &= p\,(1 + f(H)). \end{aligned} \tag{27}$$

Under very mild assumptions on the function $f(H)$ the mapping (27) will be smooth and invertible with a smooth inverse. One might assume, e.g., that $f(\cdot)$ be nonnegative and monotonically increasing for positive argument. Then, as:

$$H' =: \frac{1}{2} \left(Q^2 + P^2\right) = H\,(1 + f(H))^2, \tag{28}$$

one can solve for H and invert the mapping as:

$$q = \frac{Q}{1 + \phi(H')} \tag{29}$$

and similarly for p, with: $\phi(H') =: f(H(H'))$.

It is not hard to see that the $1D$ harmonic oscillator, whose dynamics is now given by:

$$\Gamma = P \frac{\partial}{\partial Q} - Q \frac{\partial}{\partial P}, \tag{30}$$

will be again Hamiltonian with respect to the symplectic form:

$$\omega' = dQ \wedge dP, \tag{31}$$

with H' as Hamiltonian. One can define now a new Liouville field Δ' via:

$$i_{\Delta'} \omega' = Q dP - P dQ \tag{32}$$

(hence a new linear structure) and a new complex structure:

$$J' = dP \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial P}, \tag{33}$$

and it is clear that:

$$\Gamma = J(\Delta) = J'(\Delta'). \tag{34}$$

In this way we have two different and nonlinearly related linear structures on $T^*\mathbb{R} \approx \mathbb{R}^2$¶. The $2D$ translation group $\mathbb{R}^2$ is realized then in two different ways, generated by the vector fields $\partial/\partial q$ and $\partial/\partial p$ in one case and by $\partial/\partial Q$ and $\partial/\partial P$ in the other. One interesting consequence of this is that one obtains two different ways of defining the Fourier transform[+]. In particular, when considering square-integrable functions in $L_2(\mathbb{R}^2)$, functions that are square-integrable in one coordinate system need not be so in the other, as the two Lebesgue measures are related by a non-constant Jacobian[*].

To be more explicit, let us choose the transformation

$$\begin{array}{l} q = Q(1+\lambda R^2) \\ p = P(1+\lambda R^2) \end{array}, \tag{35}$$

with $R^2 = P^2 + Q^2$, which can be inverted as

$$\begin{array}{l} Q = qK(r) \\ P = pK(r) \end{array}, \tag{36}$$

where $r^2 = p^2+q^2$, and the positive function $K(r)$ is given by the relation $R = rK(r)$ and satisies the equation $\lambda r^2 K(r)^3 + K(r) - 1 = 0$. It is not difficult to check that

$$\left| \begin{array}{c} \frac{\partial}{\partial Q} \\ \frac{\partial}{\partial P} \end{array} \right| = A \left| \begin{array}{c} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{array} \right| \tag{37}$$

where

$$\begin{aligned} A \equiv & \left| \begin{array}{cc} 1+\lambda(3Q^2+P^2) & 2\lambda PQ \\ 2\lambda PQ & 1+\lambda(Q^2+3P^2) \end{array} \right| \\ = & \left| \begin{array}{cc} 1+\lambda K(r)^2(3q^2+p^2) & 2\lambda K(r)^2 pq \\ 2\lambda K(r)^2 pq & 1+\lambda K(r)^2(q^2+3p^2) \end{array} \right| \end{aligned} \tag{38}$$

The integral curves in the plane (q, p) of the vector fields $\frac{\partial}{\partial Q}, \frac{\partial}{\partial P}$ are shown in Figure 1.

The vector fields $\frac{\partial}{\partial q}, \frac{\partial}{\partial p}$ generate the standard translation group in $\mathbb{R}^2$ associated to the linear structure:

$$\left| \begin{array}{c} q \\ p \end{array} \right| + \left| \begin{array}{c} q' \\ p' \end{array} \right| = \left| \begin{array}{c} q+q' \\ p+p' \end{array} \right| \tag{39}$$

¶ The dynamics of the harmonic oscillator will be compatible with both.

[+] Remember that the Fourier transform plays a central rôle in the implementation of the Weyl quantization scheme.

[*] Indeed: $\partial(Q,P)/\partial(q,p) = (1+f(H))(1+f(H)+2Hf'(H)) \equiv dH'/dH$.

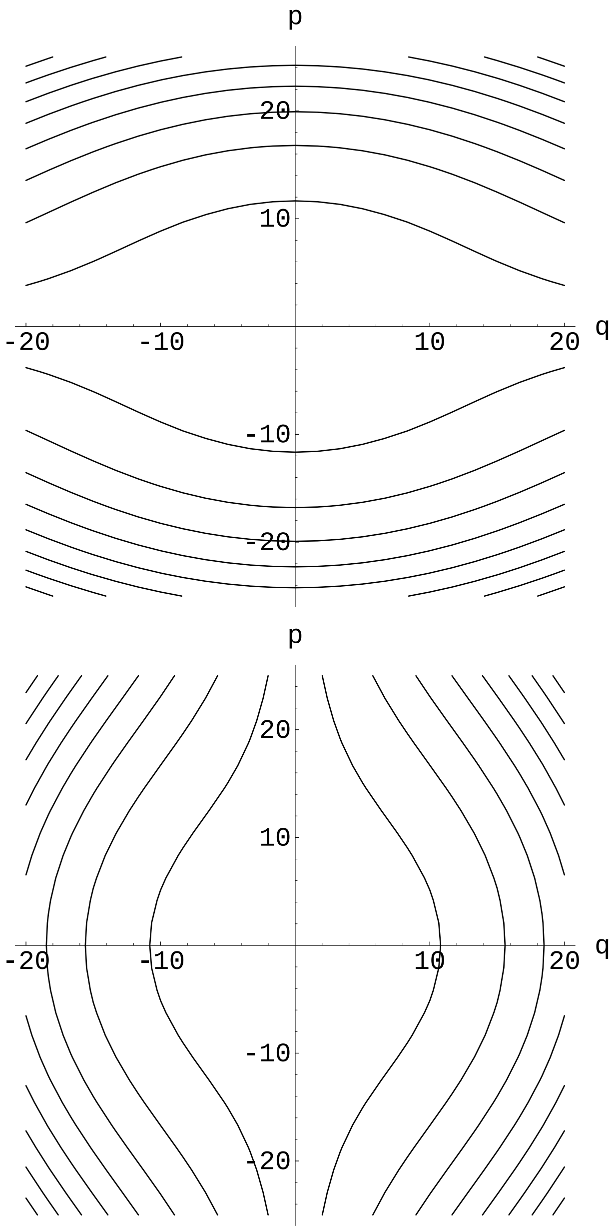

Figure 1. The integral curves in the plane (q,p) of the vector fields $\frac{\partial}{\partial Q}, \frac{\partial}{\partial P}$.

The alternative linear structure associated to the realization of the translation group by means of the vector fields $\frac{\partial}{\partial Q}, \frac{\partial}{\partial P}$ is instead given by:

$$\left|\begin{array}{c} q \\ p \end{array}\right| +_{(K)} \left|\begin{array}{c} q' \\ p' \end{array}\right| = S(r,r') \left|\begin{array}{c} K(r)q + K(r')q' \\ K(r)p + K(r')p' \end{array}\right|, \qquad (40)$$
$$S(r,r') \equiv 1 + \lambda[K(r)^2 r^2 + K(r')^2 r'^2 + 2K(r)K(r')(qq' + pp')].$$

Finally we notice also that

$$\left|\begin{array}{c} dq \\ dp \end{array}\right| = A \left|\begin{array}{c} dQ \\ dP \end{array}\right|. \qquad (41)$$

Hence the two symplectic structures $\omega = dq \wedge dp$ and $\omega' = dQ \wedge dP \equiv \omega_K$ with respect to which the vector field (34) is Hamiltonian are related by

$$\omega = D\,\omega' \;\;,\;\; D \equiv \det A \tag{42}$$

and define two different Poisson brackets, $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_K$ such that

$$\{f,g\}_K = D\,\{f,g\}. \tag{43}$$

- A second example that we shall discuss briefly here is borrowed from Quantum Mechanics, and has to do with a "superposition principle" (better, a composition rule) for pure states [8]. Given a Hilbert space $\mathcal{H}$, the space of pure states is the (complex) projective Hilbert space $\mathcal{PH}$ whose "points" are the one-dimensional projectors of the form: $\rho = |\psi\rangle\langle\psi|$, $\psi \in \mathcal{H}$, $\langle\psi|\psi\rangle = 1$. For finite-dimensional Hilbert spaces, $\mathcal{H} \approx \mathbb{C}^n$ for some n and $\mathcal{PH} \approx \mathbb{CP}^{n-1}$. In particular, for a two-level system the projective Hilbert space can be identified with the two-sphere $\mathbb{S}^2$ (the Bloch sphere). It is pretty obvious that $\mathcal{PH}$ is not a vector space. Indeed, e.g., projectors are rank-one operators, while a generic linear combination of projectors is a rank-two operator. However, one can define a rule for combining pure states as follows. Select a "fiducial" vector $|\psi_0\rangle$ in the unit sphere in the Hilbert space and the associated "fiducial" pure state $\rho_0 = |\psi_0\rangle\langle\psi_0|$. Given then any two pure states ρ_1 and ρ_2 , and assuming that they are *not* orthogonal♯ to ρ_0, we can form the linear combination:

$$|\psi\rangle = c_1\rho_1|\psi_0\rangle + c_2\rho_2|\psi_0\rangle;\; c_1, c_2 \in \mathbb{C} \tag{44}$$

and then the pure state:

$$\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = \frac{(c_1\rho_1|\psi_0\rangle + c_2\rho_2|\psi_0\rangle)\,(\bar{c}_1\langle\psi_0|\rho_1 + \bar{c}_2\langle\psi_0|\rho_2)}{\left\|c_1\rho_1|\psi_0\rangle + c_2\rho_2|\psi_0\rangle\right\|^2}. \tag{45}$$

This procedure will define a composition rule (not a linear superposition, of course) for pure states. It will depend on the choice of the fiducial vector (i.e. of ρ_0 *and* of a phase factor), but also on the linear structure of the underlying vector space. Even if $|\psi_0\rangle$ is kept fixed, alternative linear structures in $\mathcal{H}$ (or $\mathbb{C}^n$ for a system with a finite number of levels) will define therefore alternative composition rules on the projective Hilbert space.

Remark. The case of a two-level system (when $\mathcal{PH} \approx \mathbb{S}^2$) offers another interesting possibility. Indeed†† one can induce a linear structure on the punctured sphere via stereographic projection from $\mathbb{R}^2$. In this way, projecting from the "excluded" point (the projector orthogonal to ρ_0) on can induce a truly *linear* structure on the space of pure states of a two-level system, provided one excludes a single (pure) state.

♯ In the case of a two-level system, this will require excluding from the Bloch sphere the point antipodal to ρ_0.
†† See the last example discussed in the Appendix.

3. Linear Structures Associated to Regular Lagrangians

A regular Lagrangian $\mathcal{L}$ will define the symplectic structure on TQ:

$$\omega_{\mathcal{L}} = d\theta_{\mathcal{L}} = d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right) \wedge dq^i;\ \theta_{\mathcal{L}} = \left(\frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i. \tag{46}$$

We look now [9] for Hamiltonian vector fields X_j, Y^j such that:

$$i_{X_j}\omega_{\mathcal{L}} = -d\left(\frac{\partial \mathcal{L}}{\partial u^j}\right),\ i_{Y^j}\omega_{\mathcal{L}} = dq^j \tag{47}$$

which implies, of course:

$$L_{X_j}\omega_{\mathcal{L}} = L_{Y^j}\omega_{\mathcal{L}} = 0. \tag{48}$$

Explicitly:

$$i_{X_j}\omega_{\mathcal{L}} = \left(L_{X_j}\frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i - d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right)\left(L_{X_j}q^i\right) \tag{49}$$

and this implies:

$$L_{X_j}q^i = \delta^i_j,\ L_{X_j}\frac{\partial \mathcal{L}}{\partial u^i} = 0. \tag{50}$$

Similarly:

$$i_{Y^j}\omega_{\mathcal{L}} = \left(L_{Y^j}\frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i - d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right)\left(L_{Y^j}q^i\right) \tag{51}$$

and this implies in turn:

$$L_{Y^j}q^i = 0,\ L_{Y^j}\frac{\partial \mathcal{L}}{\partial u^i} = \delta^j_i. \tag{52}$$

Using then the identity:

$$i_{[Z,W]} = L_Z \circ i_W - i_W \circ L_Z, \tag{53}$$

we obtain, whenever both Z and W are Hamiltonian ($i_Z\omega_{\mathcal{L}} = dg_Z$ and similarly for W):

$$i_{[Z,W]}\omega_{\mathcal{L}} = d\left(L_Z g_W\right). \tag{54}$$

Taking now: $(Z, W) = (X_i, X_j), (X_i, Y^j)$ or (Y^i, Y^j), the Lie derivative of the Hamitonian of every field with respect to any other field is either zero or a constant (actually unity). Therefore:

$$i_{[Z,W]}\omega_{\mathcal{L}} = 0 \text{ whenever } [Z, W] = [X_i, X_j], \left[X_i, Y^j\right], \left[Y^i, Y^j\right], \tag{55}$$

which proves that:

$$[X_i, X_j] = \left[X_i, Y^j\right] = \left[Y^i, Y^j\right] = 0. \tag{56}$$

This defines an infinitesimal action of an Abelian Lie group on TQ. If this integrates to an action of the group $\mathbb{R}^{2n}$ ($\dim Q = n$) that is free and transitive, this will define a new vector space structure on TQ that is "adapted" to the Lagrangian two-form $\omega_{\mathcal{L}}$.

Spelling now explicitly eq.ns (50) and (52) we find that X_j and Y^j must be of the form:

$$X_j = \frac{\partial}{\partial q^j} + (X_j)^k \frac{\partial}{\partial u^k},\ Y^j = \left(Y^j\right)^k \frac{\partial}{\partial u^k};\ (X_j)^k, \left(Y^j\right)^k \in \mathcal{F}(TQ) \tag{57}$$

and that:

$$L_{X_j} \frac{\partial \mathcal{L}}{\partial u^i} = 0 \Rightarrow \frac{\partial^2 \mathcal{L}}{\partial u^i \partial q^j} + (X_j)^k \frac{\partial^2 \mathcal{L}}{\partial u^i \partial u^k} = 0 \tag{58}$$

and:

$$L_{Y^j} \frac{\partial \mathcal{L}}{\partial u^i} = \delta^i_j \Rightarrow \left(Y^j\right)^k \frac{\partial^2 \mathcal{L}}{\partial u^i \partial u^k} = \delta^j_i. \tag{59}$$

Therefore, the Hessian being not singular by assumption, $(Y^j)^k$ is the inverse of the Hessian matrix, while $(X_j)^k$ can be obtained algebraically from Eqn.(58).

Defining dual forms (α^i, β_i) via:

$$\alpha^i (X_j) = \delta^i_j,\ \alpha^i \left(Y^j\right) = 0 \tag{60}$$

and similarly:

$$\beta_i \left(Y^j\right) = \delta^j_i,\ \beta_i (X_j) = 0. \tag{61}$$

Testing then the identity:

$$d\theta (Z, W) = L_Z (\theta(W) - L_W (\theta (Z)) - \theta ([Z, W]) \tag{62}$$

on the pairs $(Z, W) = (X_i, X_j), (X_i, Y^j), (Y^i, Y^j)$, one proves immediately that the dual forms are all closed.■

Moreover, it is also immediate to see that:

$$\alpha^i = dq^i \tag{63}$$

and:

$$\beta_i = d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right) \tag{64}$$

and that the symplectic form can be written as:

$$\omega_{\mathcal{L}} = \beta_i \wedge \alpha^i. \tag{65}$$

Basically, what this means is that, to the extent that the definition of vector fields and dual forms is global, we have found in this way a global Darboux chart.

3.1. Examples of Adapted Linear Structures for Lagrangian Systems

- For the "standard" Lagrangian:

$$\mathcal{L} = \frac{1}{2}\delta_{ij} u^i u^j - U(q) \tag{66}$$

the solution is of course the standard one, i.e.:

$$X_j = \frac{\partial}{\partial q^j}, Y^j = \delta^{jk}\frac{\partial}{\partial u^k}. \tag{67}$$

- A particle in a (time-independent) magnetic field $\overrightarrow{B} = \nabla \times \overrightarrow{A}$. The corresponding second-order vector field is given by ($e = m = c = 1$):

$$\Gamma = u^i \frac{\partial}{\partial q^i} + \delta^{is} \epsilon_{ijk} u^j B^k \frac{\partial}{\partial u^s} \tag{68}$$

and the equations of motion are:

$$\frac{dq^i}{dt} = u^i, \ \frac{du^i}{dt} = \delta^{ir} \epsilon_{rjk} u^j B^k \ , \ i = 1, 2, 3. \tag{69}$$

The Lagrangian is given in turn by :

$$\mathcal{L} = \frac{1}{2} \delta_{ij} u^i u^j + u^i A_i. \tag{70}$$

Hence:

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial u^i} dq^i = \left(\delta_{ij} u^j + A_i \right) dq^i \tag{71}$$

and the symplectic form is‡:

$$\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}} = \delta_{ij} dq^i \wedge du^j - \frac{1}{2} \varepsilon_{ijk} B^i dq^j \wedge dq^k. \tag{72}$$

The field Γ satisfies:

$$i_\Gamma \omega_{\mathcal{L}} = dH, \tag{73}$$

with the Hamiltonian:

$$H = \frac{1}{2} \delta_{ij} u^i u^j. \tag{74}$$

Now it is easy to see that:

$$X_j = \frac{\partial}{\partial q^j} - \delta^{ik} \frac{\partial A_k}{\partial q^j} \frac{\partial}{\partial u^i}, \tag{75}$$

while:

$$Y^j = \delta^{jk} \frac{\partial}{\partial u^k}. \tag{76}$$

Dual forms $\alpha^i, \beta_i, i = 1, ..., n = \dim Q$ are defined as:

$$\begin{aligned} \left\langle \alpha^i | X_j \right\rangle &= \delta^i_j, \ \left\langle \alpha^i | Y^j \right\rangle = 0, \\ \left\langle \beta_i | X_j \right\rangle &= 0, \ \left\langle \beta_i | Y^j \right\rangle = \delta^j_i, \end{aligned} \tag{77}$$

and one finds easily:

$$\begin{aligned} \alpha^i &= dq^i, \\ \beta_i &= \delta_{ij} dU^j, \ U^j =: u^j + \delta^{jk} A_k. \end{aligned} \tag{78}$$

Notice that in this way the Cartan form (71) acquires the form:

$$\theta_{\mathcal{L}} = \pi_i dq^i, \tag{79}$$

where:

$$\pi_i = \delta_{ij} u^j + A_i, \tag{80}$$

‡ As: $\theta_{\mathcal{L}} = \theta^{(0)}_{\mathcal{L}} + A$, $\theta^{(0)}_{\mathcal{L}} = \delta_{ij} u^j dq^i$, $A = A_i dq^i$, then: $dA =: B = \frac{1}{2} \varepsilon_{ijk} B^i dq^j \wedge dq^k$, and: $\omega_{\mathcal{L}} = \omega_0 - B$.

and the symplectic form becomes:

$$\omega_{\mathcal{L}} = dq^i \wedge d\pi_i. \tag{81}$$

It appears therefore that the mapping:

$$\phi : (q, u) \to (Q, U), \tag{82}$$

with:

$$\begin{aligned} Q^i &= q^i \\ U^i &= u^i + \delta^{ik} A_k, \end{aligned} \tag{83}$$

(hence: $\pi_i = \delta_{ij} U^j$) provides us with a symplectomorphism that reduces $\omega_{\mathcal{L}}$ to the canonical form, i.e. that the chart (Q, U) is a Darboux chart "adapted" to the vector potential $\overrightarrow{A}$.
The mapping (83) is clearly invertible, and:

$$\frac{\partial q^i}{\partial Q^j} = \delta^i_j, \ \frac{\partial q^i}{\partial U^j} = 0, \tag{84}$$

while:

$$\frac{\partial u^i}{\partial U^j} = \delta^i_j, \ \frac{\partial u^i}{\partial Q^j} = -\delta^{ik} \frac{\partial A_k}{\partial Q^j}, \tag{85}$$

$A_k(q) \equiv A_k(Q)$. But then:

$$X_j = \frac{\partial}{\partial Q^j}, \ Y^j = \delta^{jk} \frac{\partial}{\partial U^k}, \tag{86}$$

as well as:

$$\alpha^i = dQ^i, \ \beta_i = d\pi_i = \delta_{ij} dU^j. \tag{87}$$

The push-forward of the Liouville field: $\Delta_0 = q^i \partial/\partial q^i + u^i \partial/\partial u^i$ will be then:

$$\Delta = \phi_* \Delta_0 = Q^i \frac{\partial}{\partial Q^i} + \left[U^i + \delta^{ik} \left(Q^j \frac{\partial A_k}{\partial Q^j} - A_k \right) \right] \frac{\partial}{\partial U^i}. \tag{88}$$

Remarks.

(i) As remarked previously: $\phi_* \Delta_0 = \Delta_0$ whenever the vector potential is homogeneous of degree one in the coordinates (constant magnetic field) an hence the mapping (83) is linear.

(ii) For an arbitrary vector potential the linear structure Δ depends on the gauge choice. This is a consequence of the mapping (83) being also gauge-dependent, which means in turn that every choice of gauge will define a *different* linear structure. The symplectic form (81) will be however gauge-independent.

(iii) Denoting collectively the old and new coordinates as (q, u) and (Q, U) respectively, eq. (83) defines a mapping:

$$(q, u) \stackrel{\phi}{\to} (Q, U). \tag{89}$$

It is then a straightforward application of the definitions (2) and (3) to show that the rules of addition and multiplication by a constant become, in this specific case:

$$(Q,U) +_{(\phi)} (Q',U') = (Q+Q', U+U' + [A(Q+Q') - (A(Q)+A(Q'))]), \tag{90}$$

and:

$$\lambda \cdot_{(\phi)} (Q,U) = (\lambda Q, \lambda U + [A(\lambda Q) - \lambda A(Q)]). \tag{91}$$

In particular, with $\lambda = e^t$, the infinitesimal version of (91) yields precisely the infinitesimal generator (88) and, if the vector potential is, as in the case of a constant magnetic field, homogeneous of degree one in the coordinates, all the terms in square brackets in eq.ns (90) and (91) vanish identically, as expected.

(iv) Notice that the origin of the new linear structure is given by: $\phi(0,0) = (0, A(0))$ and, correctly: $0 \cdot_{(\phi)} (Q,U) = (0, A(0))\ \forall (Q,U)$ as well as: $\lambda \cdot_{(\phi)} (0, A(0)) = (0, A(0))\ \forall \lambda$. Moreover: $(Q,U) + (0, A(0)) = (Q,U)\ \forall (Q,U)$. Finally, the difference between any two points (Q,U) and (Q',U') must be understood as:

$$(Q,U) -_{(\phi)} (Q',U') =: (Q,U) +_{(\phi)} \left((-1) \cdot_{(\phi)} (Q',U')\right) \tag{92}$$

and, because of: $(-1) \cdot_{(\phi)} (Q',U') = (-Q', -U' + A(Q') + A(-Q'))$, we finally get:

$$(Q,U) -_{(\phi)} (Q',U') = (Q-Q', U-U' + A(Q-Q') + A(Q') - A(Q)). \tag{93}$$

Again, if $Q' = Q, U' = U$, $(Q,U) -_{(\phi)} (Q,U) = (0, A(0))$.

If we work with the standard Euclidean metric, there is actually no need to distinguish between uppercase and lowercase indices ($Q_i =: \delta_{ij} Q^j = Q^i$ etc.). Then, the push-forward of the dynamical vector field is:

$$\widetilde{\Gamma} = \phi_* \Gamma = \left(U^i - A^i\right) \frac{\partial}{\partial Q^i} + \left(U^k - A^k\right) \frac{\partial A_k}{\partial Q^i} \frac{\partial}{\partial U^i} \tag{94}$$

and is Hamiltonian with respect to the symplectic form (81) with the Hamitonian:

$$\widetilde{H} = \phi^* H = \frac{1}{2} \delta_{ij} \left(U^i - A^i\right)\left(U^j - A^j\right). \tag{95}$$

- In particular, for a constant magnetic field $B = (0,0,B)$ with, e.g., the vector potential in the symmetric gauge:

$$\overrightarrow{A} = \frac{B}{2}\left(-q^2, q^1, 0\right) = \frac{1}{2} \overrightarrow{B} \times \overrightarrow{r},\ \overrightarrow{B} = B\widehat{k} \Rightarrow\ A_i = \frac{1}{2} \varepsilon_{ijk} B^j q^k, \tag{96}$$

$$X_1 = \frac{\partial}{\partial q^1} - \frac{B}{2}\frac{\partial}{\partial u^2},\ X_2 = \frac{\partial}{\partial q^2} + \frac{B}{2}\frac{\partial}{\partial u^1},\ X_3 = \frac{\partial}{\partial q^3}, \tag{97}$$

$$\alpha^i = dq^i \tag{98}$$

and:

$$\beta_1 = du^1 - \frac{B}{2} dq^2,\ \beta_2 = du^2 + \frac{B}{2} dq^1,\ \beta_3 = du^3, \tag{99}$$

while (see above) $\Delta = \Delta_0$, as expected.

According to eq.ns (83) and (69), the equations of motion in the new coordinates are given by:

$$\frac{d}{dt}\begin{vmatrix} Q^1 \\ Q^2 \\ U^1 \\ U^2 \end{vmatrix} = \mathbb{G}\begin{vmatrix} Q^1 \\ Q^2 \\ U^1 \\ U^2 \end{vmatrix}, \tag{100}$$

where:

$$\mathbb{G} = \left\| G^i{}_j \right\| = \begin{vmatrix} 0 & B/2 & 1 & 0 \\ -B/2 & 0 & 0 & 1 \\ -B^2/4 & 0 & 0 & B/2 \\ 0 & -B^2/4 & -B/2 & 0 \end{vmatrix}. \tag{101}$$

In other words (cfr. Eqn.(82)):

$$\begin{aligned}\phi_*\Gamma = {} & \left(U^1 + \frac{B}{2}Q^2\right)\frac{\partial}{\partial Q^1} + \left(U^2 - \frac{B}{2}Q^1\right)\frac{\partial}{\partial Q^2} \\ & + \frac{B}{2}\left(U^2 - \frac{B}{2}Q^1\right)\frac{\partial}{\partial U^1} - \frac{B}{2}\left(U^1 + \frac{B}{2}Q^2\right)\frac{\partial}{\partial U^2}.\end{aligned} \tag{102}$$

As the transformation (83) is not a point-transformation§, it comes to no surprise that the transformed vector field is no more a second-order field in the new coordinates. However, $\phi_*\Gamma$ is still Hamiltonian with respect to the symplectic form $\phi^*\omega_{\mathcal{L}} = dQ^i \wedge dU_i$ with Hamiltonian:

$$\phi^* H = \frac{1}{2}\delta_{ij}(U^i - \delta^{ik}A_k)(U^j - \delta^{jk}A_k). \tag{103}$$

Spelled out explicitly, the equations of motion in the (Q, U) coordinates are:

$$\begin{aligned}\frac{dQ^1}{dt} &= U^1 + \frac{B}{2}Q^2 \quad , \quad \frac{dQ^2}{dt} = U^2 - \frac{B}{2}Q^1, \\ \frac{dU^1}{dt} &= \frac{B}{2}\left(U^2 - \frac{B}{2}Q^1\right), \quad \frac{dU^2}{dt} = -\frac{B}{2}\left(U^1 + \frac{B}{2}Q^2\right).\end{aligned} \tag{104}$$

Hence:

$$\begin{aligned}\frac{dU^1}{dt} &= \frac{B}{2}\frac{dQ^2}{dt}, \\ \frac{dU^2}{dt} &= -\frac{B}{2}\frac{dQ^1}{dt}.\end{aligned} \tag{105}$$

Therefore:

$$\chi_1 =: U^1 - \frac{B}{2}Q^2 \text{ and: } \chi_2 = U^2 + \frac{B}{2}Q^1 \tag{106}$$

§ It is the identity on the base and acts only along the fibers.

are constants of the motion‖, and this allows an easy integration of the equations of motion. Indeed, using (106) one finds at once:

$$\begin{aligned}\frac{dQ^1}{dt} &= \chi_1 + BQ^2,\\ \frac{dQ^2}{dt} &= \chi_2 - BQ^1.\end{aligned} \tag{107}$$

and, setting:

$$Q^1(t) = \frac{\chi_2}{B} + \widetilde{Q}^1(t)\,,\; Q^2(t) = -\frac{\chi_1}{B} + \widetilde{Q}^2(t)\,, \tag{108}$$

the $\widetilde{Q}^i$'s obey the equations:

$$\frac{d\widetilde{Q}^1}{dt} = B\widetilde{Q}^2,\; \frac{d\widetilde{Q}^2}{dt} = -B\widetilde{Q}^1 \Rightarrow \frac{d^2\widetilde{Q}^i}{dt^2} + B^2\widetilde{Q}^i = 0,\; i = 1,2. \tag{109}$$

These integrate easily and, using again eq.ns (104), the final result is:

$$\left|\begin{array}{c} Q^1(t)\\ Q^2(t)\\ U^1(t)\\ U^2(t)\end{array}\right| = \mathbb{F}(t)\left|\begin{array}{c} Q^1\\ Q^2\\ U^1\\ U^2\end{array},\right| \tag{110}$$

where: $Q^1 = Q^1(0)$ etc., and $\mathbb{F}(t) =: \exp\{t\mathbb{G}\}$ is given explicitly by:

$$\mathbb{F}(t) = \left|\begin{array}{cccc} \frac{1+\cos(Bt)}{2} & \frac{\sin(Bt)}{2} & \frac{\sin(Bt)}{B} & \frac{1-\cos(Bt)}{B}\\ -\frac{\sin(Bt)}{2} & \frac{1+\cos(Bt)}{2} & \frac{\cos(Bt)-1}{B} & \frac{\sin(Bt)}{B}\\ -\frac{B\sin(Bt)}{4} & \frac{B(\cos(Bt)-1)}{4} & \frac{1+\cos(Bt)}{2} & \frac{\sin(Bt)}{2}\\ \frac{B(1-\cos(Bt))}{4} & -\frac{B\sin(Bt)}{4} & -\frac{\sin(Bt)}{2} & \frac{1+\cos(Bt)}{2}\end{array}\right|. \tag{111}$$

Checks.

That $\mathbb{F}(0) = \mathbb{I}$ can be checked by inspection. Moreover:

$$\frac{d\mathbb{F}}{dt} = \left|\begin{array}{cccc} -(B/2)\sin(Bt) & (B/2)\cos(Bt) & \cos(Bt) & \sin(Bt)\\ -(B/2)\cos(Bt) & -(B/2)\sin(Bt) & -\sin(Bt) & \cos(Bt)\\ -(B^2/4)\cos(Bt) & -(B^2/4)\sin(Bt) & -(B/2)\sin(Bt) & (B/2)\cos(Bt)\\ (B^2/4)\sin(Bt) & -(B^2/4)\cos(Bt) & -(B/2)\cos(Bt) & -(B/2)\sin(Bt)\end{array}\right|. \tag{112}$$

Hence: $(d\mathbb{F}/dt)_{t=0} = \mathbb{G}$ as it should be. That $\mathbb{F}^{-1}(d\mathbb{F}/dt) \equiv_{(t)} \mathbb{G}$ should instead be checked numerically.

Also one should check that:

$$\widetilde{\mathbb{F}} \cdot \Omega_D \cdot \mathbb{F} = \Omega_D. \tag{113}$$

This is equivalent to

$$\widetilde{\mathbb{G}} \cdot \Omega_D + \Omega_D \cdot \mathbb{G} = 0, \tag{114}$$

where:

$$\Omega_D = \left|\begin{array}{cc} \mathbf{0}_{2\times2} & \mathbb{I}_{2\times2}\\ -\mathbb{I}_{2\times2} & \mathbf{0}_{2\times2}\end{array}\right| \tag{115}$$

and this is easily checked.

‖ In fact they are proportional to the coordinates of the center of the Larmor orbit [10]. See also eq.ns (108) and (109) below.

4. Weyl Systems, Quantization and the Von Neumann Uniqueness Theorem

We recall here briefly how Weyl systems are defined and how one can implement the Weyl-Wigner-von Neumann quantization programme. Let (E, ω) be a symplectic vector space with ω a constant symplectic form. A *Weyl system* is a strongly continuous map: $\mathcal{W} : E \to \mathcal{U}(\mathcal{H})$ from E to the set of unitary operators on some Hilbert space $\mathcal{H}$ satisfying (we set here $\hbar = 1$ for simplicity):

$$\mathcal{W}(e_1)\mathcal{W}(e_2) = e^{\frac{i}{2}\omega(e_1,e_2)}\mathcal{W}(e_1+e_2);\ e_1, e_2 \in \mathcal{H} \tag{116}$$

or¶:

$$\mathcal{W}(e_1)\mathcal{W}(e_2) = e^{i\omega(e_1,e_2)}\mathcal{W}(e_2)\mathcal{W}(e_1). \tag{117}$$

It is clear that operators associated with vectors on a Lagrangian subspace will commute pairwise and can then be diagonalized simultaneously. Von Neumann's theorem states then that: *a*) Weyl systems do exist for any finite-dimensional symplectic vector space and: *b*) the Hilbert space $\mathcal{H}$ can be realized as the space of square-integrable complex functions on a Lagrangian subspace $L \subset E$ with the translationally-invariant Lebesgue measure. Decomposing then E as $L \oplus L^*$, one can define $\mathcal{U} =: \mathcal{W}|_{L^*}$ and $\mathcal{V} =: \mathcal{W}|_L$ and realize their action on $\mathcal{H} = L^2(L, d^n x)$ $(\dim E = 2n)$ as:

$$\begin{aligned}
&(\mathcal{V}(x)\psi)(y) = \psi(x+y) \\
&(\mathcal{U}(\alpha)\psi)(y) = e^{i\alpha(y)}\psi(y) \\
&x, y \in L,\ \alpha \in L^*.
\end{aligned} \tag{118}$$

As a consequence of the strong continuity of the mapping $\mathcal{W}$ one can write, using Stone's theorem [11]:

$$\mathcal{W}(e) = \exp\{i\mathcal{R}(e)\}\ \forall e \in E, \tag{119}$$

where $\mathcal{R}(e)$, which depends linearly on e, is the self-adjoint generator of the one-parameter unitary group $\mathcal{W}(te), t \in \mathbb{R}$.

If $\{\mathbb{T}(t)\}_{t\in\mathbb{R}}$ is a one-parameter group of symplectomorphisms $(\mathbb{T}(t)\mathbb{T}(t') = \mathbb{T}(t+t')\ \forall t, t'$ and: $\mathbb{T}^t(t)\omega\mathbb{T}(t) = \omega\ \forall t)$, then we can define:

$$\mathcal{W}_t(e) =: \mathcal{W}(\mathbb{T}(t)e). \tag{120}$$

This being an automorphism of the unitary group will be inner and will be therefore represented as a conjugation with a unitary transformation belonging to a one-parameter unitary group associated with the group $\{\mathbb{T}(t)\}$. If $\mathbb{T}(t)$ represents the dynamical evolution associated with a linear vector field, then we can write:

$$\mathcal{W}_t(e) = e^{it\widehat{H}}\mathcal{W}(e)e^{-it\widehat{H}} \tag{121}$$

and $\widehat{H}$ will be (again in units $\hbar = 1$) the quantum Hamiltonian of the system.

Uniqueness part of Von Neumann's theorem states that different realizations of a Weyl system on Hilberts spaces of square-integrable functions on different Lagrangian

¶ This is also called the "Weyl form" of the commutation relations.

subspaces of the same symplectic vector space are unitarily related, a conspicuous and well known example being the realization, in the case of $T^*\mathbb{R}^n$ with coordinates (q^i, p_i) and with the standard symplectic form, of the associated Weyl system on square-integrable functions of the q's or, alternatively, of the p's, that are related by the Fourier transform. In this sense the theorem is a *uniqueness* (up to unitary equivalence) theorem. We would like to stress here that it is such if the linear structure (and symplectic form) are assumed to be given once and for all.

As an example we shall consider here the case of a charged particle in a constant magnetic field [12] (and in the symmetric gauge) as described in the previous section, reinstating Planck's constant in the appropriate places. We can choose as Hilbert space that of the square-integrable functions on the Lagrangian subspace defined by: $U^i = 0, i = 1, 2$ (i.e. the subspace: $u^i = -A^i(q)$ in the original coordinates). Square-integrable wave functions will be denoted as $\psi(Q^1, Q^2)$ or $\psi(Q)$ for short. Then we can define the Weyl operators:

$$\widehat{\mathcal{W}}(x,\pi) = \exp\left\{\frac{i}{\hbar}\left[x\widehat{U} - \pi\widehat{Q}\right]\right\} =: \exp\left\{\frac{i}{\hbar}\left[x_1\widehat{U}^1 + x_2\widehat{U}^2 - \pi_1\widehat{Q}^1 - \pi_2\widehat{Q}^2\right]\right\} \tag{122}$$

acting on wavefunctions as:

$$\left(\widehat{\mathcal{W}}(x,\pi)\psi\right)(Q) = \exp\left\{-\frac{i}{\hbar}\pi\left(Q + \frac{x}{2}\right)\right\}\psi(Q + x). \tag{123}$$

Then: $\widehat{U} = -i\hbar\nabla_Q$ while $\widehat{Q}$ acts as the usual multiplication operator, i.e.: $(\widehat{Q}^i\psi)(Q) = Q^i\psi(Q)$. Eq. (122) can be rewritten in a compact way as:

$$\widehat{\mathcal{W}}(x,\pi) = \exp\left\{\frac{i}{\hbar}\xi^T \mathbf{g}\widehat{X}\right\}, \tag{124}$$

where:

$$\xi = \begin{vmatrix} x \\ \pi \end{vmatrix}, \ \widehat{X} = \begin{vmatrix} \widehat{U} \\ \widehat{Q} \end{vmatrix} \tag{125}$$

and:

$$\mathbf{g} = \begin{vmatrix} \mathbb{I}_{2\times 2} & \mathbf{0} \\ \mathbf{0} & -\mathbb{I}_{2\times 2} \end{vmatrix}. \tag{126}$$

The dynamical evolution defines then the one-parameter family of Weyl operators:

$$\widehat{\mathcal{W}}_t(x,\pi) = \widehat{\mathcal{W}}(x(t), \pi(t)) = \exp\left\{\frac{i}{\hbar}\left[x(t)\widehat{U} - \pi(t)\widehat{Q}\right]\right\} \equiv \exp\left\{\frac{i}{\hbar}\xi^T(t)\mathbf{g}\widehat{X}\right\}, \tag{127}$$

where:

$$\xi(t) = \mathbb{F}(t)\xi. \tag{128}$$

According to the standard procedure, this can be rewritten as:

$$\widehat{\mathcal{W}}_t(x,\pi) = \exp\left\{\frac{i}{\hbar}\left[x\widehat{U}(t) - \pi\widehat{Q}(t)\right]\right\} = \exp\left\{\frac{i}{\hbar}\xi^T\mathbf{g}\widehat{X}(t)\right\}, \tag{129}$$

where:

$$\begin{aligned} \widehat{X}(t) &= \widetilde{\mathbb{F}}(t)\,\widehat{X} \\ \widetilde{\mathbb{F}}(t) &= \mathbf{g}\mathbb{F}(t)^T\,\mathbf{g} \end{aligned} \tag{130}$$

and $\mathbb{F}(t)^T$ denotes the transpose of the matrix $\mathbb{F}(t)$. Explicitly:

$$\begin{aligned} \widehat{U}^1(t) &= \frac{1}{2}\widehat{U}^1(1+\cos(Bt)) - \frac{1}{2}\widehat{U}^2\sin(Bt) + \frac{B}{4}\widehat{Q}^1\sin(Bt) - \frac{B}{4}\widehat{Q}^2(1-\cos(Bt)) \\ \widehat{U}^2(t) &= \frac{1}{2}\widehat{U}^1\sin(Bt) + \frac{1}{2}\widehat{U}^2(1+\cos(Bt)) - \frac{B}{4}\widehat{Q}^1(\cos(Bt)-1) + \frac{B}{4}\widehat{Q}^2\sin(Bt), \end{aligned} \tag{131}$$

and:

$$\begin{aligned} \widehat{Q}^1(t) &= \frac{1}{B}\widehat{U}^1\sin(Bt) + \frac{1}{B}\widehat{U}^2(\cos(Bt)-1) - \frac{1}{2}\widehat{Q}^1(1+\cos(Bt)) + \frac{1}{2}\widehat{Q}^2\sin(Bt) \\ \widehat{Q}^2(t) &= \frac{1}{B}\widehat{U}^1(1-\cos(Bt)) + \frac{1}{B}\widehat{U}^2\sin(Bt) - \frac{1}{2}\widehat{Q}^1\sin(Bt) - \frac{1}{2}\widehat{Q}^2(1+\cos(Bt)). \end{aligned} \tag{132}$$

Now:

$$\widehat{\mathcal{W}_t}(x,\pi) = \widehat{\mathcal{U}}(t)^\dagger\,\widehat{\mathcal{W}}(x,\pi)\,\widehat{\mathcal{U}}(t)\,;\ \widehat{\mathcal{U}}(t) = \exp\left\{-\frac{it}{\hbar}\widehat{\mathcal{H}}\right\} \tag{133}$$

and hence:

$$\widehat{Q}^i(t) = \widehat{\mathcal{U}}(t)^\dagger\,\widehat{Q}^i\widehat{\mathcal{U}}(t) \tag{134}$$

and similarly for the $\widehat{U}^i$'s. Expanding in t we find the commutation relations:

$$\begin{aligned} \frac{i}{\hbar}\left[\widehat{U}^1,\widehat{\mathcal{H}}\right] &= \frac{B}{2}\left(\widehat{U}^2 - \frac{B}{2}\widehat{Q}^1\right) \\ \frac{i}{\hbar}\left[\widehat{U}^2,\widehat{\mathcal{H}}\right] &= -\frac{B}{2}\left(\widehat{U}^1 + \frac{B}{2}\widehat{Q}^2\right) \end{aligned} \tag{135}$$

and:

$$\begin{aligned} \frac{i}{\hbar}\left[\widehat{Q}^1,\widehat{\mathcal{H}}\right] &= -\left(\widehat{U}^1 + \frac{B}{2}\widehat{Q}^2\right) \\ \frac{i}{\hbar}\left[\widehat{Q}^2,\widehat{\mathcal{H}}\right] &= -\left(\widehat{U}^2 - \frac{B}{2}\widehat{Q}^1\right) \end{aligned} \tag{136}$$

that, using the commutation relations: $\left[\widehat{Q}^i,\widehat{U}^j\right] = i\hbar\delta^{ij}$ are consistent with the Hamiltonian:

$$\widehat{\mathcal{H}} = \frac{1}{2}\left\{\left(\widehat{U}^1 + \frac{B}{2}\widehat{Q}^2\right)^2 + \left(\widehat{U}^2 - \frac{B}{2}\widehat{Q}^1\right)^2\right\}, \tag{137}$$

which is the quantum version of the Hamiltonian (95).

In the general case, if two non-linearly related linear structures (and associated symplectic forms) are available, then one can set up two different Weyl systems realized on two different Hilbert spaces. Functions that are square-integrable in one setting need not be such in the other and viceversa, and that because, as already remarked, the Jacobian of the coordinate transformation is *not* a constant. Morover, a necessary

ingredient in the Weyl quantization program is the use of the (standard or symplectic) Fourier transform. For the same reasons as outlined above, it is also clear that, as already discussed, the two different linear structures will define genuinely *different* Fourier transforms.

In this way one can "evade" the uniqueness part of von Neumann's theorem. What the present discussion is actually meant at showing is that there are assumptions, namely that the linear structure (and symplectic form) are given once and for all and are unique, that are implicitly assumed but not explicitly stated in the usual formulations of the theorem, and that, whenever more structures are available, the situation can be much richer and lead to genuinely and non-equivalent (in the unitary sense) formulations of Quantum Mechanics.

Let us illustrate these considerations by going back to the example of the $1D$ harmonic oscillator that has been discussed in section 1. To quantize this system according to the Weyl scheme we have first of all to select a Lagrangian subspace $\mathcal{L}$ of $\mathbb{R}^2$ and a Lebesgue measure $d\mu$ on it defining then $L^2(\mathcal{L}, d\mu)$. When we endow $\mathbb{R}^2$ of the standard linear structure (39) we chose $\mathcal{L} = \{(q, 0)\}$ and $d\mu = dq$. Alternatively, when we use the linear structure (40), we take $\mathcal{L}' = \{(Q, 0)\}$ and $d\mu = dQ$. Notice that $\mathcal{L}$ and $\mathcal{L}'$ are the same subset of $\mathbb{R}^2$, defined by the conditions $P = p = 0$ and with the coordinates related by the relation $Q = qK(r = |q|)$. Nevertheless the two Hilbert spaces $L^2(\mathcal{L}, d\mu)$ and $L^2(\mathcal{L}', d\mu')$ are not related via a unitary map since the Jacobian of the coordinate transformations is not constant: $d\mu = (1 + 3\lambda Q^2)d\mu'$.

As a second step in the Weyl scheme, we construct in $L^2(\mathcal{L}, d\mu)$ the operator $\hat{U}(\alpha)$:

$$\left(\hat{U}(\alpha)\psi\right)(q) = e^{i\alpha q/\hbar}\psi(q) \ , \ \psi(q) \in L^2(\mathcal{L}, d\mu), \tag{138}$$

whose generator is $\hat{x} = q$, and the operator $\hat{V}(h)$:

$$\left(\hat{V}(h)\psi\right)(q) = \psi(q + h) \ \psi(q) \in L^2(\mathcal{L}, d\mu), \tag{139}$$

which is generated by $\hat{\pi} = -i\hbar\partial/\partial q$, and implements the translations defined by the linear structure (39). The quantum Hamiltonian can be written as $H = \hbar\left(a^\dagger a + \frac{1}{2}\right)$ where $a = (\hat{x} + i\hat{\pi})/\sqrt{2}\hbar$ (here the adjoint is taken with respect to the complex structure defined by the Lebesgue measure dq).
Similar expressions hold in $L^2(\mathcal{L}', d\mu')$ for $\hat{x}'$, $\hat{\pi}'$ and $\hat{U}'(\alpha)$, $\hat{V}'(h)$. Notice that, as seen as operators in the former Hilbert space, $\hat{V}'(h)$ implements translations with respect to the linear structure (40):

$$(\hat{V}'(h)\psi)(q) = \psi(q +_{(K)} h). \tag{140}$$

Now the quantum Hamiltonian is $H' = \hbar\left(A^{\dagger\prime} A + \frac{1}{2}\right)$ with $A = (\hat{x}' + i\hat{\pi}')/\sqrt{2}\hbar$, where now the adjoint is taken with respect to the complex structure defined by the Lebesgue measure dQ^+.

+ A direct calculation shows that $a^\dagger = \frac{1}{\sqrt{2}\hbar}\left(q - \hbar\frac{\partial}{\partial q}\right)$ whereas $a^{\dagger\prime} = \frac{1}{\sqrt{2}\hbar}\left[q - \hbar\frac{\partial}{\partial q} - \frac{6\hbar\lambda K(r)q}{(1+3\lambda K(r)^2 q^2)^2}\right]$. Also $A^{\dagger\prime} = \frac{1}{\sqrt{2}\hbar}\left[K(r)q - \hbar(1 + 3\lambda K(r)^2 q^2)\frac{\partial}{\partial q}\right]$. We notice that the transformation relating $a^\dagger$ and $A^{\dagger\prime}$ is not of the type considered in in the second reference of [7].

Finally we recall that, following the Weyl-Wigner-Moyal program [13], one defines an "inverse" mapping of (actually Hilbert-Schmidt) operators onto square-integrable functions in phase space endowed with a non-commutative "$*$-product", the Moyal product [14]. The Moyal product is defined as:

$$(f * g)(q,p) = f(q,p) \exp\left\{ \frac{i\hbar}{2} \left[\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \right] \right\} g(q,p) . \tag{141}$$

It defines in turn the Moyal bracket:

$$\{f,g\}_M =: \frac{1}{i\hbar}(f * g - g * f) \tag{142}$$

and:

$$\{f,g\}_M = \{f,g\}_\omega + \mathcal{O}\left(\hbar^2\right), \tag{143}$$

where $\{.,\}_\omega$ is the Poisson bracket defined by the symplectic form ω, and similarly with the use of the second (i.e., $(\partial/\partial Q, \partial/\partial P)$) linear structure. Different (and not unitarily equivalent) Weyl systems will lead to different Moyal products and brackets, and to different Poisson brackets in the classical limit.

For example, in the case of the $1D$ harmonic oscillator one has Eqn. (141) for the ordinary Moyal product and,

$$(f *_K g)(Q,P) = f(Q,P) \exp\left\{ \frac{i\hbar}{2} \left[\frac{\overleftarrow{\partial}}{\partial Q} \frac{\overrightarrow{\partial}}{\partial P} - \frac{\overleftarrow{\partial}}{\partial P} \frac{\overrightarrow{\partial}}{\partial Q} \right] \right\} g(Q,P), \tag{144}$$

which define the corresponding Moyal brackets $\{f,g\}_M$ and $\{f,g\}_{M_K}$. It is not difficult to check that, since

$$\left[\frac{\overleftarrow{\partial}}{\partial Q} \frac{\overrightarrow{\partial}}{\partial P} - \frac{\overleftarrow{\partial}}{\partial P} \frac{\overrightarrow{\partial}}{\partial Q} \right] = D \left[\frac{\overleftarrow{\partial}}{\partial Q} \frac{\overrightarrow{\partial}}{\partial P} - \frac{\overleftarrow{\partial}}{\partial P} \frac{\overrightarrow{\partial}}{\partial Q} \right], \tag{145}$$

eq. (43) will hold in the classical limit $\hbar \to 0$.

Appendix A. Further examples of "exported" linear structures

- Relativistic addition of velocities. Let $E = \mathbb{R}$, $M = (-1,1)$ and:

$$\phi : x \to X =: \tanh x. \tag{A.1}$$

Then:

$$\lambda \cdot_{(\phi)} X = \tanh\left(\lambda \tanh^{-1}(X)\right) \tag{A.2}$$

and:

$$\begin{aligned} \lambda \cdot_{(\phi)} \left(\lambda' \cdot_{(\phi)} X\right) &= \lambda \cdot_{(\phi)} \tanh\left(\lambda' \tanh^{-1}(X)\right) = \\ &= \tanh\left(\lambda\lambda' \tanh^{-1}(X)\right) = (\lambda\lambda') \cdot_{(\phi)} X, \end{aligned} \tag{A.3}$$

while:

$$X +_{(\phi)} Y = \tanh\left(\tanh^{-1}(X) + \tanh^{-1}(Y)\right) = \frac{X+Y}{1+XY}, \tag{A.4}$$

which is nothing but the one-dimensional relativistic law (in appropriate units) for the addition of velocities. It is also simple to prove that:

$$\begin{aligned}\left(X +_{(\phi)} Y\right) +_{(\phi)} Z &= \tanh\left(\tanh^{-1}\left(X +_{(\phi)} Y\right) + \tanh^{-1}(Z)\right) = \\ &= \tanh\left(\tanh^{-1} X + \tanh^{-1}(Y) + \tanh^{-1}(Z)\right)\end{aligned} \tag{A.5}$$

i.e. that:

$$\left(X +_{(\phi)} Y\right) +_{(\phi)} Z = X +_{(\phi)} \left(Y +_{(\phi)} Z\right). \tag{A.6}$$

Explicitly:

$$X +_{(\phi)} Y +_{(\phi)} Z = \frac{X + Y + Z + XYZ}{1 + XY + XZ + YZ}. \tag{A.7}$$

The mapping (9) is now:

$$X(t) = \tanh\left(e^t \tanh^{-1}(X)\right) \tag{A.8}$$

and we obtain, for the Liouville field on $(-1, 1)$:

$$\Delta(X) = \left(1 - X^2\right) \tanh^{-1}(X) \frac{\partial}{\partial X} \tag{A.9}$$

and $\Delta(X) = 0$ for $X = 0$.

- Another similar example involves $E = \mathbb{R}, M = \mathbb{R}^+ = (0, +\infty)$ and:

$$\phi = x \to X = \exp(x). \tag{A.10}$$

Then one can see easily that:

$$X \cdot_{(\phi)} X' = XX' \tag{A.11}$$

and:

$$\lambda \cdot_{(\phi)} X = X^\lambda. \tag{A.12}$$

In this way:

$$X(t) = \phi\left(e^t \phi^{-1}(X)\right) = X^{e^t} = \exp\left[e^t \ln(X)\right] \tag{A.13}$$

and one finds the "adapted" Liouville field:

$$\Delta(X) = X \ln(X) \frac{\partial}{\partial X}. \tag{A.14}$$

Notice that here the fixed point of the Liouville field is $X = 1 = \phi(0)$.

- (In this example ϕ is a homeomorphism and not a diffeomorphism). Let $E = M = \mathbb{R}$ and:

$$\phi : x \to X = x^3. \tag{A.15}$$

Then:

$$X +_{(\phi)} Y = \left(\sqrt[3]{X} + \sqrt[3]{Y}\right)^3 \tag{A.16}$$

and:

$$\lambda \cdot_{(\phi)} X = \lambda^3 X. \tag{A.17}$$

The proof that (4) and (5) are satisfied is elementary and will be omitted.

- As a last example, we can consider the inverse stereographic projection of $\mathbb{R}^2$ onto the Riemann sphere:

$$\Phi : \mathbb{R}^2 \to \mathbb{S}^2 - \{0, 0, 1\} \tag{A.18}$$

by:

$$\Phi : z \to X = \{x_1, x_2, x_3\}, x_1 = \frac{z + \overline{z}}{|z|^2 + 1}, x_2 = \frac{z - \overline{z}}{i(|z|^2 + 1)}, x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}, \tag{A.19}$$

that inverts to:

$$z = \Phi^{-1}(X) = \frac{x_1 + ix_2}{1 - x_3}. \tag{A.20}$$

Multiplication by a constant results in:

$$\lambda \cdot_{(\Phi)} X = X' \tag{A.21}$$

with $X' = \{x'_1, x'_2, x'_3\}$ and:

$$x'_1 = \frac{2\lambda x_1}{\lambda^2 + 1 + x_3(\lambda^2 - 1)}, x'_2 = \frac{2\lambda x_1}{\lambda^2 + 1 + x_3(\lambda^2 - 1)}, x'_3 = \frac{\lambda^2 - 1 + x_3(\lambda^2 + 1)}{\lambda^2 + 1 + x_3(\lambda^2 - 1)}, \tag{A.22}$$

$(x'_1)^2 + (x'_2)^2 + (x'_3)^2 = 1$. Again: $X' = X$ for $\lambda = 1$, while, for $\lambda = 0, X' = \{0, 0, -1\}$. Moreover, for $\lambda \to \infty$, $x'_{1,2} \to 0$ while: $x'_3 \to 1$. Setting then: $\lambda = e^t$ and taking derivatives, we obtain:

$$\frac{dx_1}{dt}|_{t=0} = -x_1x_3, \frac{dx_2}{dt}|_{t=0} = -x_2x_3 \tag{A.23}$$

and:

$$\frac{dx_3}{dt} = 1 - x_3^2. \tag{A.24}$$

This defines the vector field:

$$\Delta = -x_1x_3\frac{\partial}{\partial x_1} - x_2x_3\frac{\partial}{\partial x_2} + \left(1 - x_3^2\right)\frac{\partial}{\partial x^3} \tag{A.25}$$

and it is easy to check that:

$$\mathcal{L}_\Delta\left((x_1)^2 + (x_2)^2 + (x_3)^2\right) = 0, \tag{A.26}$$

i.e. that Δ is indeed tangent to the sphere.
Switching to spherical polar coordinates:

$$x_1 = \sin\theta\cos\phi, x_2 = \sin\theta\sin\phi, x_3 = \cos\theta, \tag{A.27}$$

the equations of motion (A.23,A.24) take the simple form:

$$\frac{d\theta}{dt} = -\sin\theta, \frac{d\phi}{dt} = 0 \tag{A.28}$$

and hence the field (A.25) become simply:

$$\Delta = -\sin\theta\frac{\partial}{\partial\theta}, \tag{A.29}$$

which has a fixed point at $\theta = \pi$.

Explicitly, eq.ns (A.28) integrate to:

$$\tan\left(\frac{\theta(t)}{2}\right) = \tan\left(\frac{\theta}{2}\right) e^{-t}, \phi(t) = const. = \phi \tag{A.30}$$

$(\theta(0) = \theta, \phi(0) = \phi)$, and θ flows towards the "North Pole" $\theta = 0$ when $t \to +\infty$ and towards the "South Pole" $\theta = \pi$ when $t \to -\infty$.

Polar coordinates make all the calculations easier. Representing z as: $z = \rho \exp(i\varphi)$ and X with the polar angles θ and ϕ, the maps (A.19) and (A.20) become simply:

$$\begin{aligned} &\phi = \varphi \\ &\sin\theta = \frac{2\rho}{\rho^2+1}, \cos\theta = \frac{\rho^2-1}{\rho^2+1} \end{aligned} \tag{A.31}$$

and:

$$\begin{aligned} &\varphi = \phi \\ &\rho = \cot(\theta/2) \end{aligned} \tag{A.32}$$

respectively.
Then, given $X = (\theta, \phi), X' = (\theta', \phi')$:

$$\Phi^{-1}(X) + \Phi^{-1}(X') = \rho \exp(i\varphi), \tag{A.33}$$

with:

$$\rho = \sqrt{\cot^2(\theta/2) + \cot^2(\theta'/2) + 2\cot(\theta/2)\cot(\theta'/2)\cos(\phi - \phi')} \tag{A.34}$$

and:

$$\cos\varphi = \frac{\cot(\theta/2)\cos\phi + \cot(\theta'/2)\cos\phi'}{\rho}, \sin\varphi = \frac{\cot(\theta/2)\sin\phi + \cot(\theta'/2)\sin\phi'}{\rho}. \tag{A.35}$$

It follows that:

$$X +_{(\Phi)} X' = (\theta, \phi) +_{(\Phi)} (\theta', \phi') = (\theta'', \phi''), \tag{A.36}$$

where $\phi'' = \varphi$, with φ given by Eqn.(A.35) and:

$$\sin\theta'' = \frac{2\rho}{\rho^2+1}, \cos\theta'' = \frac{\rho^2-1}{\rho^2+1}, \tag{A.37}$$

with ρ given now by Eqn.(A.34). In particular, if $\phi = \phi'$, then $\phi'' = \phi$ and:

$$\begin{aligned} &\sin\theta'' = 2\frac{\cot(\theta/2) + \cot(\theta'/2)}{(\cot(\theta/2) + \cot(\theta'/2))^2 + 1}, \\ &\cos\theta'' = \frac{(\cot(\theta/2) + \cot(\theta'/2))^2 - 1}{(\cot(\theta/2) + \cot(\theta'/2))^2 + 1}. \end{aligned} \tag{A.38}$$

Concerning multiplication by a (real) constant, we have:

$$\lambda \cdot_{(\Phi)} X = \lambda \cdot_{(\Phi)} (\theta, \phi) = (\theta', \phi), \tag{A.39}$$

with:

$$\sin\theta' = \frac{2\lambda\cot(\theta/2)}{\lambda^2\cot^2(\theta/2) + 1}, \cos\theta' = \frac{\lambda^2\cot^2(\theta/2) - 1}{\lambda^2\cot^2(\theta/2) + 1}. \tag{A.40}$$

Here too, for $\lambda \to 0, +\infty$, θ' flows towards the South Pole and the North Pole respectively.

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